PHYSICAL REVIEW E **66**, 041105 (2002)

Solvable *n*-species aggregation processes with joint annihilation

Jianhong Ke* and Zhenquan Lin†
Department of Physics, Wenzhou Normal College, Wenzhou 325027, China

(Received 16 May 2002; published 14 October 2002)

We study the kinetic behavior of the aggregation-annihilation processes of an n-species ($n \geqslant 3$) system, in which an irreversible aggregation reaction occurs between any two clusters of the same species and an irreversible complete annihilation reaction occurs only between one certain A^n species and each of the other A^m species ($m=1,2,\ldots,n-1$). Based on the mean-field theory, we investigate the rate equations of the processes to obtain the asymptotic solutions of the cluster-mass distributions in several different cases. The results show that the evolution behavior of the system depends crucially on the ratios of the equivalent aggregation rate of A^m species and the aggregation rate of A^n species to the annihilation rate. The cluster-mass distribution of each species always obeys a conventional scaling law or a modified one, and the scaling exponents depend only on the reaction rates for most cases. However, when both the equivalent aggregation rate of A^m species and the aggregation rate of A^n species are twice as large as the annihilation rate, the scaling exponents depend on the reaction rates as well as the initial concentrations.

DOI: 10.1103/PhysRevE.66.041105 PACS number(s): 82.20.−w, 68.43.Jk, 05.70.Ln, 89.75.Da

I. INTRODUCTION

Aggregation is a fundamental kinetic phenomenon and underlies a wide variety of nonequilibrium processes in nature, such as colloid science, aerosol formation, droplet growth, and so on [1–4]. Typically, irreversible aggregation processes are described by the reaction scheme $A_i+A_j \to A_{i+j}$. Here, A_i denotes a cluster consisting of i monomers. The clusters A_i and A_j can bond spontaneously and result in a larger cluster A_{i+j}. The kinetics of these aggregation processes has been well understood since the 1970s [4–12]. In the past decade, much interest was also devoted to understanding the kinetics of another basic bimolecular annihilation reaction $A+B \to$ product [13–17], which stemmed from the discovery of its fluctuation-dominated kinetics [18,19]. The examples of this reaction scheme are electron-hole, soliton-antisoliton, and defect-antidefect recombination processes. Recently, Krapivsky first investigated the competition between aggregation and annihilation processes of a two-species system, in which an irreversible aggregation reaction occurs only between any two clusters of the same species and an annihilation reaction occurs between the different species [20]. In a spirit close to this work, many studies were also focused on the competition between aggregation and annihilation processes [21–25]. It is of interest that for some special initial cases the evolution behaviors of the aggregates may obey a scaling law in the long-time limit. In these works, there are two types of annihilation reaction schemes. The first one is the partial annihilation, where the larger cluster is conserved with the monomer difference number of the two reactants after the reaction [20,21,24]. The second one is the complete annihilation, where the binary annihilation between the two different species always produces the inert aggregate independent of the reactant masses [23,25]. However, there are a few studies concerning the aggregation-annihilation system with multispecies. Zhang and Yang found that the evolution of an aggregation-annihilation close-chain model with multispecies is quite different from that of an open-chain model [24]. In our aggregation-annihilation chain model, it was found that the evolution behavior of the system depends crucially on the number of the species [26]. These indicate that the kinetics of the multispecies cases may be very rich.

The goal of this work is to investigate the competition between aggregation and complete annihilation processes of n types of species, A^l ($l=1,2,\ldots,n$, $n \geqslant 3$). We assume that an irreversible aggregation occurs between any two clusters of the same species, $A_i^l+A_j^l \xrightarrow{K_l(i,j)} A_{i+j}^l$, and an irreversible complete annihilation reaction occurs simultaneously between the clusters of the different species, $A_i^l+A_j^{l'} \xrightarrow{J_{ll'}(i,j)}$ inert ($l,l'=1,2,\ldots,n, l\neq l'$), where A_i^l denotes a cluster consisting of i-mers of A^l species. The rate of the aggregation between A_i^l and A_j^l clusters equals $K_l(i,j)$, and that of annihilation between A_i^l and $A_j^{l'}$ clusters is $J_{ll'}(i,j)$. In general, the annihilation reaction may occur between any two different species [23,25]. In this model, we consider an interesting simple case: the complete annihilation reaction occurs only between a certain A^n species and any other A^m species ($m=1,2,\ldots,n-1$), namely, A^n clusters are annihilated jointly by all the other A^m clusters.

We study our model in the mean-field limit. The mean-field assumption neglects the spatial fluctuation of the reactant densities and therefore typically applies to the case in which the spatial dimension d of the system is equal to or greater than a critical dimension d_c [20,23]. The investigation of the aggregation process can also be based on the particle coalescence model (PCM) in the diffusion-controlled limit [7,23]. For pure aggregation processes, the PCM found that $d_c=2$ [7]. For the general aggregation-annihilation system (here, complete annihilation occurs between any two different species), Ben-Naim and Krapivsky proposed that the critical dimension is $d_c=2$ and also confirmed the mean-

---

*Electronic address: kejianhong@163.com
†Electronic address: linzhenquan@yahoo.com.cn

field predictions above this critical dimension numerically [23]. Since our model interpolates between the pure aggregation and the general aggregation-annihilation processes, it is natural to expect that for our model the critical dimension is the same, $d_c = 2$. On the other hand, Sokolov and Blumen also found that the marginal dimension for the irreversible two-species coagulation model with partial annihilation is 2 [21]. This may confirm the above expectation of the critical dimension of our model, although for the pure *n*-species annihilation processes the critical dimension is $d_c = 4(n-1)/(2n-3)$ [27]. We defer the numerical confirmation of the critical dimension of our system to a future work. In our investigation, we assume that the spatial dimension *d* of our system is greater than 2 and therefore the mean-field theory is valid.

We believe that our aggregation-annihilation model may mimic a wide variety of physical, biological, and social phenomena. For example, in a multicomponent chemical system, the aggregations of the clusters of the same kind produce open-chain polymers, which have energetic reactive edges and therefore can continue participating in the reaction processes. The annihilation reactions between two different species produce inert closed polymers, which lose their reactive edges and then withdraw from the reaction processes. On the other hand, we assume that only a certain species has the cohesive affinity for all the others while the others exclude one another, thus only certain species can cohere with each of the other species and then produce the inert closed polymers, namely, the annihilation reactions occur only between the certain species and any other species. In the social sciences, an army fighting with its allied forces against the enemies in the war may be regarded as a joint annihilation. It is also believed that the irreversible multispecies aggregation-annihilation processes are of interest in studying the scaling properties of their evolution behaviors. According to this work, it is found that the evolution kinetics of the system depends strongly on the ratios of the aggregation rates to the annihilation rate. Meanwhile, the initial concentrations also play an important role in some special cases. We observe unusual behaviors of this model in several different cases, such as the breakdown of the conventional scaling law, modified scaling with nonuniversal exponents, and so on.

The paper is organized as follows. In Sec. II, we describe an irreversible *n*-species ($n \geqslant 3$) aggregation-annihilation model with constant reaction rates and give the corresponding mean-field rate equations. Then we derive the asymptotic solutions of the cluster-mass distributions in several different cases as illustrated in Table I. Finally, a brief summary is given in Sec. III.

II. MODEL OF *n*-SPECIES AGGREGATION PROCESSES WITH JOINT ANNIHILATION

In our investigation, the theoretical approach to the aggregation processes is based on the mean-field rate equations, which assumes that the reaction proceeds with a rate proportional to the reactant concentrations. In this *n*-species model, the concentrations of A^l clusters of *k*-mers are denoted as a_{lk}, $l = 1,2,\dots,n$. For simplicity, we consider a model with constant reaction rates, and set all the annihilation reaction rates equal to *J*. In order to investigate thoroughly the kinetic evolution of this irreversible aggregation-annihilation system, we assume that the aggregation rates of A^l species have different constant values I_l. By generalizing the rate equations of the aggregation-annihilation processes given by Ben-Naim and Krapivsky [23], we then write out the governing rate equations for this system

TABLE I. Organization of Sec. II, where I_n is the aggregation rate of A^n species, J is the annihilation rate, and I_e is the equivalent aggregation rate of A^m species $(m = 1,2\dots,n-1)$.

Case	Title of section
A	$I_n > 2J$ and $I_e < 2J$
B	$I_n > 2J$ and $I_e = 2J$
C	$I_n > 2J$ and $I_e > 2J$
D	$I_n = 2J$ and $I_e < 2J$
E	$I_n = I_e = 2J$
F	$I_n = 2J$ and $I_e > 2J$
G	$I_n < 2J$ and $I_e \geqslant 2J$
H	$I_n < 2J$ and $I_e < 2J$

$$\frac{da_{mk}}{dt} = I_m\left(\frac{1}{2}\sum_{i+j=k} a_{mi}a_{mj} - a_{mk}\sum_{j=1}^{\infty} a_{mj}\right) - Ja_{mk}\sum_{j=1}^{\infty} a_{nj}, \quad m = 1,2,\dots,n-1,$$

$$\frac{da_{nk}}{dt} = I_n\left(\frac{1}{2}\sum_{i+j=k} a_{ni}a_{nj} - a_{nk}\sum_{j=1}^{\infty} a_{nj}\right) - Ja_{nk}\sum_{m=1}^{n-1}\sum_{j=1}^{\infty} a_{mj}. \tag{1}$$

As we aim to find out the analytical solutions of the evolution behaviors of the clusters and investigate their long-time scaling properties, we assume that for each species there only exist the monomer clusters at $t = 0$ and the concentrations of A^l monomers are equal to A_{l0}, $l = 1,2,\dots,n$. Then the monodisperse initial conditions are

$$a_{lk}(0) = A_{l0}\delta_{k1}, \quad l = 1,2,\dots,n. \tag{2}$$

Under the above initial conditions, Eqs. (1) can be solved by the help of ansatz [20]

$$a_{lk}(t) = A_l(t)[a_l(t)]^{k-1}, \quad l = 1,2,\dots,n. \tag{3}$$

Substituting Eq. (3) into Eqs. (1), we transform them into the following differential equations:

$$\frac{da_m}{dt} = \frac{I_m A_m}{2}, \quad \frac{dA_m}{dt} = -\frac{I_m A_m^2}{1-a_m} - \frac{JA_mA_n}{1-a_n},$$

$$m = 1,2,\dots,n-1,$$

$$\frac{da_n}{dt}=\frac{I_nA_n}{2},\quad \frac{dA_n}{dt}=-\frac{I_nA_n^2}{1-a_n}-JA_n\sum_{m=1}^{n-1}\frac{A_m}{1-a_m}. \tag{4}$$

Correspondingly, the initial conditions of Eqs. (4) are

$$a_l=0,\ A_l=A_{l0},\quad l=1,2,\ldots,n \ \text{ at } \ t=0. \tag{5}$$

Introducing new variables $\alpha_l(t)$,

$$\alpha_l=(1-a_l)^{-1},\quad l=1,2,\ldots,n, \tag{6}$$

we can recast the differential equations (4) to the following equations:

$$\frac{d^2\alpha_m}{dt^2}=-\frac{2J}{I_n\alpha_n}\frac{d\alpha_m}{dt}\frac{d\alpha_n}{dt},$$

$$\frac{d^2\alpha_n}{dt^2}=-\frac{d\alpha_n}{dt}\sum_{m=1}^{n-1}\left(\frac{2J}{I_m\alpha_m}\frac{d\alpha_m}{dt}\right), \tag{7}$$

where $m=1,2,\ldots,n-1$. The corresponding initial conditions become

$$\alpha_l=1,\ \frac{d\alpha_l}{dt}=\frac{I_lA_{l0}}{2},\quad l=1,2,\ldots,n \text{ at } t=0. \tag{8}$$

One can determine the following integrals of Eqs. (7):

$$\frac{d\alpha_m}{dt}=\frac{I_mA_{m0}}{2}\alpha_n^{-2J/I_n},\quad \frac{d\alpha_n}{dt}=\frac{I_nA_{n0}}{2}\prod_{m=1}^{n-1}\alpha_m^{-2J/I_m}, \tag{9}$$

where $m=1,2,\ldots,n-1$. From Eqs. (9), we then obtain

$$\frac{1}{I_mA_{m0}}(\alpha_m-1)=\frac{1}{I_1A_{10}}(\alpha_1-1), \tag{10}$$

$$\left(\frac{d\alpha_m}{dt}\right)^{-I_n/2J}\frac{d^2\alpha_m}{dt^2}=-JA_{n0}\left(\frac{I_mA_{m0}}{2}\right)^{-I_n/2J}\prod_{m'=1}^{n-1}\times\alpha_{m'}^{-2J/I_{m'}}\frac{d\alpha_m}{dt},$$

$$m=1,2,\ldots,n-1. \tag{11}$$

Substituting Eq. (10) into Eq. (11), we derive the following equations:

$$\left(\frac{d\alpha_m}{dt}\right)^{-I_n/2J}\frac{d^2\alpha_m}{dt^2}=-JA_{n0}\left(\frac{I_mA_{m0}}{2}\right)^{-I_n/2J}\prod_{m'=1}^{n-1}\left(1-\frac{I_{m'}A_{m'0}}{I_mA_{m0}}+\frac{I_{m'}A_{m'0}}{I_mA_{m0}}\alpha_m\right)^{-2J/I_{m'}}\frac{d\alpha_m}{dt},\quad m=1,2,\ldots,n-1. \tag{12}$$

For convenience, we set $\Sigma_{1\leqslant m\leqslant n-1}1/I_m=1/I_e$. Here, I_e can be considered as the equivalent aggregation rate of all A^m species $(m=1,2,\ldots,n-1)$. Then we discuss the solutions of Eqs. (9) in several different cases.

A. $I_n>2J$ and $I_e<2J$ case

We assume that the system reaches its steady state at $t\rightarrow\infty$ and its conditions are then given as follows:

$$\frac{da_l}{dt}=\frac{1}{\alpha_l^2}\frac{d\alpha_l}{dt}=0,\quad \frac{dA_l}{dt}=\frac{2}{I_l\alpha_l^2}\frac{d^2\alpha_l}{dt^2}-\frac{4}{I_l\alpha_l^3}\left(\frac{d\alpha_l}{dt}\right)^2=0,\quad l=1,2,\ldots,n. \tag{13}$$

Thus we can conclude that either $\alpha_l\rightarrow\infty$ or $d\alpha_l/dt\rightarrow 0$ at $t\rightarrow\infty$. From Eq. (12), we know that for the case of $I_n>2J$, $\alpha_m\rightarrow\infty$ $(m=1,2,\ldots,n-1)$ at $t\rightarrow\infty$. Thus, $\alpha_m\gg 1$ at $t\gg 1$. In the long-time limit, Eq. (12) reduces to the following asymptotic equation:

$$\left(\frac{d\alpha_m}{dt}\right)^{-I_n/2J}\frac{d^2\alpha_m}{dt^2}\simeq -JA_{n0}\left(\frac{I_mA_{m0}}{2}\right)^{-I_n/2J}\prod_{m'=1}^{n-1}\left(\frac{I_{m'}A_{m'0}}{I_mA_{m0}}\right)^{-2J/I_{m'}}\alpha_m^{-2J/I_e}\frac{d\alpha_m}{dt},\quad m=1,2,\ldots,n-1. \tag{14}$$

One can then obtain the integral of Eq. (14) in the case of $I_e<2J$,

$$\frac{d\alpha_m}{dt}\simeq\left(c_{1m}-c_{2m}\alpha_m^{1-2J/I_e}\right)^{2J/(2J-I_n)}\simeq c_{1m}^{2J/(2J-I_n)},$$

$$m=1,2,\ldots,n-1, \tag{15}$$

where c_{1m} and c_{2m} are integral constants. In the long-time limit, the asymptotic solution of $\alpha_m(t)$ can be derived from Eqs. (12) as follows:

$$\alpha_m(t)\simeq C_{1,m}t,\quad m=1,2,\ldots,n-1, \tag{16}$$

where

$$C_{1,m}=\left[\frac{(I_n-2J)A_{n0}}{2}\left(\frac{I_mA_{m0}}{2}\right)^{-I_n/2J}\int_1^\infty\prod_{1\leqslant m'\leqslant n-1}\left(1-\frac{I_{m'}A_{m'0}}{I_mA_{m0}}+\frac{I_{m'}A_{m'0}}{I_mA_{m0}}x\right)^{-2J/I_{m'}}dx+\left(\frac{I_mA_{m0}}{2}\right)^{1-I_n/2J}\right]^{2J/(2J-I_n)}.$$

In the symmetrical initial case, $I_mA_{m0}=I_1A_{10}$ for all $m=1,2,\ldots,n-1$, $C_{1,m}$ can be given in explicit form

$$C_{1,m}=\left[\frac{(I_n-2J)I_eA_{n0}}{2(2J-I_e)}(I_mA_{m0}/2)^{-I_n/2J}+(I_mA_{m0}/2)^{1-I_n/2J}\right]^{2J/(2J-I_n)}.$$

Substituting Eq. (16) into Eqs. (9), we derive the asymptotic solution of $\alpha_n(t)$ in the long-time limit

$$\alpha_n(t)\simeq C_{1,n}-C_{2,n}t^{1-2J/I_e},\tag{17}$$

where $C_{1,n}=(I_1A_{10}/2C_{1,1})^{I_n/2J}$ and $C_{2,n}=[I_nI_eA_{n0}/2(2J-I_e)]\Pi_{m=1}^{n-1}C_{1,m}^{-2J/I_m}$. We then obtain the asymptotic solutions of the cluster-mass distributions of A^m and A^n species in the long-time limit,

$$a_{mk}(t)\simeq\frac{2}{I_mC_{1,m}}t^{-2}(1-C_{1,m}^{-1}t^{-1})^{k-1},\quad m=1,2,\ldots,n-1,$$

$$a_{nk}(t)\simeq\frac{2(2J-I_e)C_{2,n}}{I_nI_eC_{1,n}^2}t^{-2J/I_e}\left(1-\frac{1}{C_{1,n}-C_{2,n}t^{1-2J/I_e}}\right)^{k-1}.\tag{18}$$

Further, the mass distributions of the larger clusters can be rewritten in the following forms:

$$a_{mk}\simeq\frac{2}{I_mC_{1,m}}t^{-2}\exp(-x_m),\quad m=1,2,\ldots,n-1,\tag{19}$$

$$a_{nk}\simeq\frac{2(2J-I_e)C_{2,n}}{I_nI_eC_{1,n}^2}\left(\frac{C_{1,n}-1}{C_{1,n}}\right)^k t^{-2J/I_e}\exp(-x_n),\tag{20}$$

which are valid in the regions $k\gg1$, $t\gg1$, $x_m=(k/C_{1,m})t^{-1}=\text{finite}$ and $x_n=[C_{2,n}/C_{1,n}(C_{1,n}-1)]kt^{-(2J-I_e)/I_e}=\text{finite}$.

Equation (19) indicates that for the monodisperse initial cases the cluster-mass distribution of A^m species comes in a scaling regime. In general irreversible aggregation processes, the average mass of the clusters increases indefinitely with time. Thus, one may assume that this typical mass plays a role analogous to that of correlation length in ordinary critical phenomena. Introducing such characteristic mass $S(t)$, one can write the mass distribution of the clusters in the following conventional scaling form [20]:

$$c_k(t)\simeq t^{-w}\Phi[k/S(t)],\quad S(t)\propto t^z.\tag{21}$$

The total number $N(t)$ and the total mass $M(t)$ of the clusters can also be expressed in the power-law forms [20]

$$N(t)=\sum_{k=1}^\infty c_k(t)\propto t^{-\lambda},\quad M(t)=\sum_{k=1}^\infty kc_k(t)\propto t^{-\mu}.\tag{22}$$

It is not difficult to obtain the scaling exponent relations

$$\lambda=w-z,\quad \mu=w-2z.\tag{23}$$

For A^m species $(m=1,2,\ldots,n-1)$, the scaling function is exponential, $\Phi(x)=\exp(-x)$, and the typical mass is $S(t)\sim t$. From Eq. (19) we obtain all these exponents for A^m species,

$$w=2,\quad z=1,\quad \lambda=1,\quad \mu=0,\tag{24}$$

which are independent of the reaction rates and of the initial concentrations.

From Eq. (20), we find that the conventional scaling description (21) of the cluster-mass distribution breaks down for A^n species. One can modify the conventional scaling description (21) as follows [20]:

$$c_k(t)\simeq b^kt^{-w}\Phi[k/S(t)],\quad S(t)\propto t^z,\tag{25}$$

where b is a constant and $0<b<1$. Correspondingly, the exponent relations (23) become

$$\lambda=w,\quad \mu=w.\tag{26}$$

Thus, we obtain the exponents for A^n species

$$w=\lambda=\mu=\frac{2J}{I_e},\quad z=\frac{2J-I_e}{I_e},\tag{27}$$

which are not universal constants and depend on the equivalent aggregation rate of A^m species $(m=1,2,\ldots,n-1)$ and on the annihilation rate J. The modified scaling form (25) indicates that two different mass scales are associated with A^n species. One is a growing scale $S(t)\sim t^{(2J-I_e)/I_e}$, which is forced by A^m species. Another is a time-independent scale $S=C_{1,n}$, which dominates the behavior of the A^n species in the long-time limit. Ben-Naim and Krapivsky also found that this nonuniversal phenomenon exists in the irreversible aggregation-annihilation processes [20,23].

In this case, we find that the total cluster number of each species decreases with time because $\lambda>0$ for all species. Moreover, comparison between the total mass

$$M_m(t)\simeq\frac{2C_{1,m}}{I_m}\quad\text{for } A^m \text{ species},\quad m=1,2,\ldots,n-1,$$

$$M_n(t)\simeq\frac{2(2J-I_e)C_{2,n}}{I_nI_e}t^{-2J/I_e}\quad\text{for } A^n \text{ species}\tag{28}$$

shows that all the A^m species can survive at $t \to \infty$ while A^n species is annihilated completely by A^m species. This is independent of the initial data of A_{l0} $(l=1,2,\ldots,n)$. For example, even in the case with $A_{n0} > \Sigma_{1 \leqslant m \leqslant n-1} A_{m0}$, the rate of light A^n clusters with small size bonding to form the large ones is greater than that of the corresponding A^m clusters because of $I_n > I_e$, and therefore the large clusters of A^n species are readily annihilated by the light clusters of A^m species. This implies that the mass decay rate of A^n species caused by the complete annihilation is greater than that of A^m species. In this case, the results indicate that A^m species $(m=1,2,\ldots,n-1)$ can survive in the end.

B. $I_n > 2J$ and $I_e = 2J$ case

In this case, we also find that $\alpha_m \gg 1$ $(m=1,2,\ldots,n-1)$ at $t \gg 1$. We can obtain the asymptotic integral of Eq. (12) as follows:

$$\frac{d\alpha_m}{dt} \simeq C_{3,m} (\ln \alpha_m)^{2J/(2J-I_n)}, \quad m=1,2,\ldots,n-1, \tag{29}$$

where

$$C_{3,m} = \left[\frac{(I_n-2J)A_{n0}}{2} \left(\frac{I_m A_{m0}}{2} \right)^{-I_n/2J} \prod_{m'=1}^{n-1} \times \left(\frac{I_{m'} A_{m'0}}{I_m A_{m0}} \right)^{-2J/I_{m'}} \right]^{2J/(2J-I_n)}.$$

We integrate Eq. (29) and then derive the solution of $\alpha_m(t)$ at $t \gg 1$ in the implicit form

$$\sum_{i=1}^{\infty} \left[(-1)^i \prod_{j=1}^{i} [I_n/(I_n-2J)-j] \alpha_m (\ln \alpha_m)^{-i+2J/(I_n-2J)} \right] + \alpha_m (\ln \alpha_m)^{2J/(I_n-2J)} \simeq C_{3,m} t, \quad m=1,2,\ldots,n-1. \tag{30}$$

Here, if there exists an integral N meeting the equation $2J/(I_n-2J)-N=0$, the infinite sum in Eq. (30) will reduce to the finite terms of $i=1 \sim N$. In the long-time limit, the summation in Eq. (30) is by far smaller than $\alpha_m (\ln \alpha_m)^{2J/(I_n-2J)}$ and is then negligible. Thus Eq. (30) reduces to

$$\alpha_m (\ln \alpha_m)^{2J/(I_n-2J)} \simeq C_{3,m} t, \quad m=1,2,\ldots,n-1. \tag{31}$$

In the long-time limit, the asymptotic solution of $\alpha_m(t)$ can be given as

$$\alpha_m(t) \simeq C_{3,m} t (\ln t)^{2J/(2J-I_n)}, \quad m=1,2,\ldots,n-1. \tag{32}$$

Substituting Eq. (32) into Eqs. (9), we determine the asymptotic solution of $\alpha_n(t)$,

$$\alpha_n(t) \simeq C_{3,n} (\ln t)^{I_n/(I_n-2J)}, \tag{33}$$

where $C_{3,n} = [(I_n-2J)A_{n0}/2] \Pi_{m=1}^{n-1} C_{3,m}^{-2J/I_m}$. Thus we obtain the asymptotic descriptions of the cluster-mass distributions for all the species,

$$a_{mk}(t) \simeq \frac{2}{I_m C_{3,m}} t^{-2} (\ln t)^{-2J/(2J-I_n)} \exp(-x_m),$$

$$m=1,2,\ldots,n-1,$$

$$a_{nk}(t) \simeq \frac{2}{(I_n-2J)C_{3,n}} t^{-1} (\ln t)^{-(2I_n-2J)/(I_n-2J)} \exp(-x_n), \tag{34}$$

which are valid in the scaling regions $k \gg 1$, $t \gg 1$; $x_m = (k/C_{3,m}) t^{-1} (\ln t)^{-2J/(2J-I_n)} = \text{finite}$ and $x_n = (k/C_{3,n}) \times (\ln t)^{-I_n/(I_n-2J)} = \text{finite}$.

Equations (34) indicate that for this case the conventional scaling description (21) breaks down for each species. The evolution of cluster-mass distribution of any species has rather peculiar scaling behavior. We can modify the above scaling description (21) further into [25]

$$c_k(t) \simeq C_0 h^k [g(t)]^{-w_1} [f(t)]^{-w_2} \Phi[k/S(t)],$$

$$S(t) \propto [g(t)]^{z_1} [f(t)]^{z_2}, \quad g'(t),\ f'(t) > 0, \tag{35}$$

where C_0 and h denote two constants, and $0 < h \leqslant 1$. $g(t)$ and $f(t)$ are unusual functions of time, such as e^t, $\ln t$, 2^t, and so on. The total number and the total mass of the clusters can be rewritten as

$$N(t) = \sum_{k=1}^{\infty} c_k(t) \propto [g(t)]^{-\lambda_1} [f(t)]^{-\lambda_2},$$

$$M(t) = \sum_{k=1}^{\infty} k c_k(t) \propto [g(t)]^{-\mu_1} [f(t)]^{-\mu_2}. \tag{36}$$

From Eqs. (35) and (36), we obtain the following exponent relations:

$$\lambda_1 = w_1 - z_1, \quad \mu_1 = w_1 - 2z_1, \quad \lambda_2 = w_2 - z_2,$$

$$\mu_2 = w_2 - 2z_2 \quad \text{for} \quad h=1;$$

$$\lambda_1 = \mu_1 = w_1, \quad \lambda_2 = \mu_2 = w_2 \quad \text{for} \quad 0 < h < 1. \tag{37}$$

In this case, we find all the scaling exponents by letting $g(t)=t$ and $f(t)=\ln t$:

$$w_1 = 2, \quad w_2 = \frac{2J}{2J-I_n}, \quad z_1 = 1, \quad z_2 = \frac{2J}{2J-I_n}, \quad \lambda_1 = 1, \quad \lambda_2 = 0,$$

$$\mu_1 = 0, \quad \mu_2 = \frac{2J}{I_n-2J} \quad \text{for } A^m \text{ clusters},$$

$$m=1,2,\ldots,n-1,$$

$$w_1 = 1, \quad w_2 = \frac{2I_n-2J}{I_n-2J}, \quad z_1 = 0, \quad z_2 = \frac{I_n}{I_n-2J},$$

$$\lambda_1=1,\quad \lambda_2=1,$$

$$\mu_1=1,\quad \mu_2=\frac{2J}{2J-I_n}\quad \text{for } A^n \text{ clusters,} \tag{38}$$

which indicate that the exponents depend on the aggregation rate of A^n species and on the annihilation rate J. Comparison between the total mass of A^m species $(m=1,2,\ldots,n-1)$ and that of A^n species, $M_m(t)\propto(\ln t)^{-2J/(I_n-2J)}$ and $M_n(t)\propto t^{-1}(\ln t)^{2J/(I_n-2J)}$, shows that A^m clusters always dominate over A^n clusters in the long-time limit. This is independent of the initial concentrations. In this case, both the total number and the total mass of each species decrease with time, and no species can survive at $t\rightarrow\infty$.

C. $I_n>2J$ and $I_e>2J$ case

For this case $\alpha_m\gg1$ $(m=1,2,\ldots,n-1)$ at $t\gg1$, which is similar to that in Sec. II A. Thus, we can obtain the asymptotic integral of Eq. (12) as follows:

$$\frac{d\alpha_m}{dt}\simeq C_{3,m}[I_e/(I_e-2J)]^{2J/(2J-I_n)}\alpha_m^{2J(I_e-2J)/I_e(2J-I_n)},$$

$$m=1,2,\ldots,n-1. \tag{39}$$

In the long-time limit, the asymptotic solution of $\alpha_m(t)$ is then obtained,

$$\alpha_m(t)\simeq C_{4,m}t^{I_e(I_n-2J)/(I_eI_n-4J^2)},\quad m=1,2,\ldots,n-1, \tag{40}$$

where

$$C_{4,m}=[(I_eI_n-4J^2)C_{3,m}/I_e(I_n-2J)]^{I_e(I_n-2J)/(I_eI_n-4J^2)}$$
$$\times[I_e/(I_e-2J)]^{2JI_e/(4J^2-I_eI_n)}.$$

Substituting Eq. (40) into Eqs. (9), we determine the following asymptotic solution of $\alpha_n(t)$:

$$\alpha_n(t)\simeq C_{4,n}t^{I_n(I_e-2J)/(I_eI_n-4J^2)}, \tag{41}$$

where $C_{4,n}=[(I_eI_n-4J^2)A_{n0}/2(I_e-2J)]\Pi_{m=1}^{n-1}C_{4,m}^{-2J/I_m}$. Thus, we obtain the conventional scaling descriptions of the cluster-mass distributions in the long-time limit,

$$a_{mk}(t)\simeq\frac{2I_e(I_n-2J)}{I_m(I_eI_n-4J^2)C_{4,m}}t^{-(2I_eI_n-2I_eJ-4J^2)/(I_eI_n-4J^2)}\exp(-x_m),\quad m=1,2,\ldots,n-1,$$

$$a_{nk}(t)\simeq\frac{2(I_e-2J)}{(I_eI_n-4J^2)C_{4,n}}t^{-(2I_eI_n-2I_nJ-4J^2)/(I_eI_n-4J^2)}\exp(-x_n), \tag{42}$$

with scaling variables

$$x_m=C_{4,m}^{-1}t^{-I_e(I_n-2J)/(I_eI_n-4J^2)},$$

$$x_n=C_{4,n}^{-1}t^{-I_n(I_e-2J)/(I_eI_n-4J^2)}. \tag{43}$$

The scaling exponents are then obtained,

$$w=\frac{2I_eI_n-2I_eJ-4J^2}{I_eI_n-4J^2},\quad z=\frac{I_eI_n-2I_eJ}{I_eI_n-4J^2},\quad \lambda=1,$$

$$\mu=\frac{2I_eJ-4J^2}{I_eI_n-4J^2}\quad \text{for } A^m \text{ clusters,}\quad m=1,2,\ldots,n-1,$$

$$w=\frac{2I_eI_n-2I_nJ-4J^2}{I_eI_n-4J^2},\quad z=\frac{I_eI_n-2I_nJ}{I_eI_n-4J^2},\quad \lambda=1,$$

$$\mu=\frac{2I_nJ-4J^2}{I_eI_n-4J^2}\quad \text{for } A^n \text{ clusters.} \tag{44}$$

The results show that both the total number and the total mass of each species decrease with time because $\lambda,\mu>0$ for each species. Hence, no species can survive at $t\rightarrow\infty$. In this case, the exponents depend on the annihilation and aggregation rates. When $I_e>I_n$, the value of μ for A^n species is less than that for A^m species, thus A^n clusters dominate over the corresponding clusters of the other species in the long-time limit. This is independent of initial concentrations. When $I_e<I_n$, it is quite the contrary to the case of $I_e>I_n$. In the special case of $I_e=I_n$, all the species have the same scaling evolution of the cluster-mass distribution.

D. $I_n=2J$ and $I_e<2J$ case

It is obvious that for this case $\alpha_m\gg1$ $(m=1,2,\ldots,n-1)$ at $t\gg1$. Thus, we obtain the following asymptotic integral of Eq. (12) in the long-time limit:

$$\frac{d\alpha_m}{dt}\simeq c_{3m}\exp\left[\frac{2I_eJA_{n0}}{(2J-I_e)I_mA_{m0}}\prod_{m'=1}^{n-1}\left(\frac{I_{m'}A_{m'0}}{I_mA_{m0}}\right)^{-2J/I_{m'}}\right.$$
$$\left.\times\alpha_m^{1-2J/I_e}\right],\quad m=1,2,\ldots,n-1, \tag{45}$$

where c_{3m} are integral constants. Since $\alpha_m^{1-2J/I_e}\ll1$ at $t\gg1$, we obtain

$$\frac{d\alpha_m}{dt} \simeq c_{3m}, \quad m=1,2,\ldots,n-1. \tag{46}$$

From Eqs. (9) and (12), we then determine the asymptotic solutions of $\alpha_m(t)$ and $\alpha_n(t)$ in the long-time limit,

$$\alpha_m(t) \simeq C_{5,m} t, \quad m=1,2,\ldots,n-1,$$

$$\alpha_n(t) \simeq C_{5,n} - C_{6,n} t^{1-2J/I_e}, \tag{47}$$

where

$$C_{5,m} = \frac{I_m A_{m0}}{2} \exp\left[-\frac{2JA_{n0}}{I_m A_{m0}} \int_1^{\infty} \prod_{1 \leqslant m' \leqslant n-1} \times \left(1 - \frac{I_{m'} A_{m'0}}{I_m A_{m0}} + \frac{I_{m'} A_{m'0}}{I_m A_{m0}} x \right)^{-2J/I_{m'}} dx \right],$$

$$C_{5,n} = I_1 A_{10}/2C_{5,1},$$

and

$$C_{6,n} = [I_e I_n A_{n0}/2(2J - I_e)] \prod_{1 \leqslant m \leqslant n-1} C_{5,m}^{-2J/I_m}.$$

Under the symmetrical initial conditions, $I_m A_{m0} = I_1 A_{10}$ for $m=1,2,\ldots,n-1$, the constant $C_{5,m}$ has the explicit value $C_{5,m} = (I_m A_{m0}/2)\exp[2I_e J A_{n0}/(2J - I_e) I_m A_{m0}]$. It is shown that the asymptotic solutions (47) of this case are similar to Eqs. (16) and (17) in Sec. II A. By substituting $C_{1,l}$ ($l=1,2,\ldots,n$) and $C_{2,n}$ in Eqs. (19) and (20) with $C_{5,l}$ and $C_{6,n}$, respectively, we can obtain the modified equations (19) and (20) of the cluster-mass distributions for this case. So the results of this case are identical with those in Sec. II A.

E. $I_n = 2J$ and $I_e = 2J$ case

Since $\alpha_m \gg 1$ ($m=1,2,\ldots,n-1$) at $t \gg 1$, we derive the following asymptotic integral of Eq. (12) for this case:

$$\frac{d\alpha_m}{dt} \simeq \frac{I_m A_{m0}}{2} \alpha_m^{-\beta}, \quad m=1,2,\ldots,n-1, \tag{48}$$

where $\beta = 2JA_{n0} \Pi_{1 \leqslant m' \leqslant n-1} (I_{m'} A_{m'0})^{-2J/I_{m'}}$. From Eq. (48), we obtain the asymptotic solution of $\alpha_m(t)$ in the long-time limit,

$$\alpha_m(t) \simeq C_{7,m} t^{\gamma}, \quad m=1,2,\ldots,n-1, \tag{49}$$

where $\gamma = (1+\beta)^{-1}$ and $C_{7,m} = (I_m A_{m0}/2\gamma)^{\gamma}$. The solution of $\alpha_n(t)$ can then be determined from Eqs. (9) as follows:

$$\alpha_n(t) \simeq C_{7,n} t^{1-\gamma}, \tag{50}$$

where $C_{7,n} = I_1 A_{10}/2\gamma C_{7,1}$. In the long-time limit, we obtain the conventional scaling solutions of the cluster-mass distributions

$$a_{mk} \simeq \frac{2\gamma}{I_m C_{7,m}} t^{-1-\gamma} \exp(-x_m), \quad x_m = C_{7,m}^{-1} k t^{-\gamma},$$

$$m=1,2,\ldots,n-1,$$

$$a_{nk} \simeq \frac{2(1-\gamma)}{I_n C_{7,n}} t^{-2+\gamma} \exp(-x_n), \quad x_n = C_{7,n}^{-1} k t^{-1+\gamma}. \tag{51}$$

For this case, the scaling exponents are

$$w = 1+\gamma, \quad z = \gamma, \quad \lambda = 1, \quad \mu = 1-\gamma$$

$$\text{for all the } A^m \text{ clusters}, \quad m=1,2,\ldots,n-1,$$

$$w = 2-\gamma, \quad z = 1-\gamma, \quad \lambda = 1, \quad \mu = \gamma \quad \text{for } A^n \text{ clusters}. \tag{52}$$

The results show that the scaling exponents w, z, and μ are dependent both on the reaction rates and on the initial concentrations. Moreover, when $2JA_{n0} > \Pi_{1 \leqslant m \leqslant n-1} (I_m A_{m0})^{2J/I_m}$, the value of μ for each of A^m species ($m=1,2,\ldots,n-1$) is larger than that for A^n species and therefore A^n clusters may dominate over A^m clusters at $t \gg 1$, and it is quite the contrary in the reverse case of $2JA_{n0} < \Pi_{1 \leqslant m \leqslant n-1} (I_m A_{m0})^{2J/I_m}$. When $2JA_{n0} = \Pi_{1 \leqslant m \leqslant n-1} (I_m A_{m0})^{2J/I_m}$, all the species have the same constant exponents $w=3/2$, $z=1/2$, $\lambda=1$, and $\mu=1/2$. In this case, we also find that both the total number and the total mass of any species decrease with time. Hence, no species can survive in the end.

F. $I_n = 2J$ and $I_e > 2J$ case

In this case, we can obtain the same asymptotic integral (45) from Eq. (12) in the long-time limit. We integrate Eq. (45) and then derive the solution of $\alpha_m(t)$ in the implicit form

$$\sum_{i=1}^{\infty} \left[(-1)^{i+1} C_{8,m}^{-i} \prod_{j=0}^{i-1} \left(\frac{I_e}{I_e - 2J} - j \right) \alpha_m^{2iJ/I_e - i + 1} \exp(C_{8,m} \alpha_m^{1-2J/I_e}) \right] \simeq c_{3m} t, \quad m=1,2,\ldots,n-1, \tag{53}$$

where

$$C_{8,m}=[2I_eJA_{n0}/(I_e-2J)I_mA_{m0}]\prod_{1\leqslant m'\leqslant n-1}\times(I_{m'}A_{m'0}/I_mA_{m0})^{-2J/I_{m'}}.$$

Here, if there exists an integral M satisfying the equation $2MJ/I_e-M+1=0$, the infinite terms in Eq. (53) would reduce to the finite terms of $i=1\sim M$. In the long-time limit, Eq. (53) can be approximately rewritten as

$$C_{8,m}^{-1}\frac{I_e}{I_e-2J}\alpha_m^{2J/I_e}\exp(C_{8,m}\alpha_m^{1-2J/I_e})\simeq c_{3m}t,\quad m=1,2,\dots,n-1. \tag{54}$$

We then determine the asymptotic solution of $\alpha_m(t)$ from Eq. (54) as follows:

$$\alpha_m(t)\simeq C_{8,m}^{I_e/(2J-I_e)}(\ln t)^{I_e/(I_e-2J)},\quad m=1,2,\dots,n-1. \tag{55}$$

Substituting Eq. (55) into Eqs. (9), one can obtain the solution of $\alpha_n(t)$ in the long-time limit,

$$\alpha_n(t)\simeq C_{8,m}t(\ln t)^{2J/(2J-I_e)}, \tag{56}$$

where $C_{8,n}=[I_1(I_e-2J)A_{10}/I_e]C_{8,1}^{I_e/(I_e-2J)}$. In this case, the conventional scaling description (21) also breaks down for each species, and we obtain the peculiar scaling descriptions for the mass distributions as follows:

$$a_{mk}(t)\simeq\frac{2I_e}{I_m(I_e-2J)}C_{8,m}^{I_e/(I_e-2J)}t^{-1}\times(\ln t)^{-(2I_e-2J)/(I_e-2J)}\exp(-x_m),$$

$$x_m=C_{8,m}^{I_e/(I_e-2J)}k(\ln t)^{-I_e/(I_e-2J)},\quad m=1,2,\dots,n-1,$$

$$a_{nk}(t)\simeq\frac{2}{I_nC_{8,n}}t^{-2}(\ln t)^{-2J/(2J-I_e)}\exp(-x_n),$$

$$x_n=\frac{k}{C_{8,n}}t^{-1}(\ln t)^{-2J/(2J-I_e)}. \tag{57}$$

By letting $g(t)$ and $f(t)$ in Eq. (35) equal to t and $\ln t$, respectively, we obtain the scaling exponents for this case,

$$w_1=1,\ \ w_2=\frac{2I_e-2J}{I_e-2J},\ \ z_1=0,\ \ z_2=\frac{I_e}{I_e-2J},$$

$$\lambda_1=1,\ \ \lambda_2=1,$$

$$\mu_1=1,\quad \mu_2=\frac{2J}{2J-I_e}\quad \text{for } A^m \text{ clusters},$$

$$m=1,2,\dots,n-1,$$

$$w_1=2,\quad w_2=\frac{2J}{2J-I_e},\quad z_1=1,\quad z_2=\frac{2J}{2J-I_e},$$

$$\lambda_1=1,\quad \lambda_2=0,$$

$$\mu_1=0,\quad \mu_2=\frac{2J}{I_e-2J}\quad \text{for } A^n \text{ clusters}. \tag{58}$$

These indicate that the exponents depend on the equivalent aggregation I_e of A^m species and on the annihilation rate J. In this case, A^n species dominates over the A^m species in the long-time limit, which is just contrary to that in Sec. II B. Moreover, both the total number and the total mass of each species decrease with time. Hence, no species can survive in the end.

G. $I_n<2J$ and $I_e\geqslant 2J$ case

We can conclude from Eq. (12) that for this case $\alpha_m\to C_{9,m}$ ($C_{9,m}$ are finite constants, $m=1,2,\dots,n-1$) at $t\to\infty$ and the implicit expressions for $C_{9,m}$ are

$$\frac{I_mA_{m0}}{(2J-I_n)A_{n0}}=\int_1^{C_{9,m}}\prod_{m'=1}^{n-1}\left(1-\frac{I_{m'}A_{m'0}}{I_mA_{m0}}+\frac{I_{m'}A_{m'0}}{I_mA_{m0}}x\right)^{-2J/I_{m'}}dx,\quad m=1,2,\dots,n-1. \tag{59}$$

For the special case of $I_mA_{m0}=I_1A_{10}$ ($m=1,2,\dots,n-1$), one can easily derive the explicit expressions for $C_{9,m}$ as follows:

$$C_{9,m}=\exp[I_mA_{m0}/(2J-I_n)A_{n0}]\quad \text{for } I_e=2J,$$

$$C_{9,m}=[I_m(I_e-2J)A_{m0}/I_e(2J-I_n)A_{n0}+1]^{I_e/(I_e-2J)}\quad \text{for } I_e>2J. \tag{60}$$

From Eqs. (9), we obtain the following asymptotic solution of $\alpha_n(t)$ in the long-time limit:

$$\alpha_n(t)\simeq C_{9,n}t, \tag{61}$$

where $C_{9,n}=(I_nA_{n0}/2)\Pi_{1\leqslant m\leqslant n-1}C_{9,m}^{-2J/I_m}$. Substituting Eq. (61) into Eqs. (9), we determine the asymptotic solution of $\alpha_m(t)$ as follows:

TABLE II. Summary of the results in different cases.

Case	Summary of the results
$I_n \geqslant 2J$	(i) A^m species ($m=1,2,\ldots,n-1$) obey the conventional scaling law (21);
	(ii) A^n species obeys the modified scaling law (25).
$I_e < 2J$	(iii) All A^m species can survive in the end.
$I_n > 2J$	(i) All species obey the further modified scaling law (35);
	(ii) A^m clusters dominate over A^n clusters in the long-time limit.
$I_e = 2J$	(iii) No species can survive in the end.
$I_n > 2J$	(i) All species obey the conventional scaling law (21);
	(ii) the governing exponents depend on the reaction rates.
$I_e > 2J$	(iii) No species can survive in the end.
$I_n = 2J$	(i) All species obey the conventional scaling law (21);
	(ii) the exponents depend on reaction rates as well as initial concentrations.
$I_e = 2J$	(iii) No species can survive in the end.
$I_n = 2J$	(i) All species obey the modified scaling law (35);
	(ii) A^n clusters dominate over A^m clusters in the long-time limit.
$I_e > 2J$	(iii) No species can survive in the end.
$I_n < 2J$	(i) All A^m species obey the modified scaling law (25);
	(ii) A^n species obeys the conventional scaling law (21).
$I_e \geqslant 2J$	(iii) Only A^n species can survive in the end.
$I_n < 2J$	(i) If $\eta > 1$, this case is equivalent to the case of $I_n \geqslant 2J$ and $I_e < 2J$;
	(ii) if $\eta = 1$, this case is equivalent to the case of $I_n > 2J$ and $I_e > 2J$.
$I_e < 2J$	(iii) If $\eta < 1$, this case is equivalent to the case of $I_n < 2J$ and $I_e \geqslant 2J$.

$$\alpha_m(t) \simeq C_{9,m} - C_{10,m} t^{1-2J/I_n}, \quad m=1,2,\ldots,n-1, \tag{62}$$

where $C_{10,m} = [I_m I_n A_{m0}/2(2J-I_n)] C_{9,n}^{-2J/I_n}$. In the long-time limit, the conventional scaling description of the cluster-mass distribution for A^n species is then obtained,

$$a_{nk}(t) \simeq \frac{2}{I_n C_{9,n}} t^{-2} \exp(-x_n), \quad x_n = \frac{k}{C_{9,n}} t^{-1}, \tag{63}$$

with the scaling exponents (24). The conventional description (21) breaks down for A^m species ($m=1,2,\ldots,n-1$) and its mass distribution satisfies the modified scaling description

$$a_{mk}(t) \simeq \frac{2(2J-I_n)C_{10,m}}{I_m I_n C_{9,m}^2} \left(\frac{C_{9,m}-1}{C_{9,m}} \right)^k t^{-2J/I_n} \exp(-x_m),$$

$$x_m = \frac{C_{10,m}}{C_{9,m}(C_{9,m}-1)} k t^{-(2J-I_n)/I_n}, \tag{64}$$

with the exponents

$$w = \lambda = \mu = \frac{2J}{I_n}, \quad z = \frac{2J-I_n}{I_n}. \tag{65}$$

These indicate that the exponents for A^m species ($m=1,2,\ldots,n-1$) depend on the annihilation rate J as well as the aggregation rate of A^n species, while those for A^n species are constant values independent of the reaction rates. In this case, the mass scale $S(t)$ of A^n species grows as t. For all A^m species, their growing scales are the same, $S(t) \sim t^{(2J-I_n)/I_n}$, which is forced by A^n species, and their time-independent scales are different values, $S'_m = C_{9,m}$, which are dependent on the details of the reaction conditions. Moreover, the results also imply that A^n clusters dominate over A^m clusters in the long-time limit and only A^n clusters can survive in the end.

H. $I_n < 2J$ and $I_e < 2J$ case

In this case, it is difficult to determine the exact solutions of the cluster-mass distributions. Now, we consider a special case, i.e., $I_m A_{m0} = I_1 I_{10}$ for $m=1,2,\ldots,n-1$. We conclude from Eq. (10) that $\alpha_m(t) = \alpha_1(t)$, $m=1,2,\ldots,n-1$. Equations (9) then reduce to

$$\frac{d\alpha_1}{dt} = \frac{I_1 A_{10}}{2} \alpha_n^{-2J/I_n}, \quad \frac{d\alpha_n}{dt} = \frac{I_n A_{n0}}{2} \alpha_1^{-2J/I_e}. \tag{66}$$

We can obtain the following equations from Eqs. (66)

$$\frac{d\alpha_1}{dt} = [b_1 \alpha_1^{-(2J-I_e)/I_e} + b_2]^{2J/(2J-I_n)}, \tag{67}$$

$$\frac{d\alpha_n}{dt} = [b_3 \alpha_n^{-(2J-I_n)/I_n} + b_4]^{2J/(2J-I_e)}, \tag{68}$$

where

$$b_1 = [(2J-I_n) I_e A_{n0}/2(2J-I_e)] (I_1 A_{10}/2)^{-I_n/2J},$$

$$b_2 = \{[(2J-I_e) I_1 A_{10} - (2J-I_n) I_e A_{n0}]/2(2J-I_e)\} \times (I_1 A_{10}/2)^{-I_n/2J},$$

$$b_3 = [(2J-I_e)I_nI_1A_{10}/2(2J-I_n)I_e](I_nA_{n0}/2)^{-I_e/2J},$$

and

$$b_4 = \{I_n[(2J-I_n)I_eA_{n0}-(2J-I_e)I_1A_{10}]/2(2J-I_n)I_e\} \times (I_nA_{n0}/2)^{-I_e/2J}.$$

When $(2J-I_e)I_1A_{10} > (2J-I_n)I_eA_{n0}$, $b_2 > 0$, and $b_4 < 0$. Under these conditions, $\alpha_1 \to \infty$ and $\alpha_n \to C$ (C is a finite constant) at $t \to \infty$. In the long-time limit, Eq. (67) can be approximately rewritten as

$$\frac{d\alpha_1}{dt} \simeq C_{11,1}, \tag{69}$$

where $C_{11,1} = b_2^{2J/(2J-I_n)}$. This directly yields

$$\alpha_1(t) \simeq C_{11,1}t. \tag{70}$$

One can then determine the asymptotic solution of $\alpha_n(t)$ in the long-time limit,

$$\alpha_n(t) \simeq C_{11,n} - C_{12,n}t^{1-2J/I_e}, \tag{71}$$

where $C_{11,n} = (I_1A_{10}/2C_{11,1})^{I_n/2J}$ and $C_{12,n} = [I_eI_nA_{n0}/2(2J-I_e)]C_{11,1}^{-2J/I_e}$. Equations (70) and (71) are similar to Eqs. (16) and (17) in Sec. II A, respectively. Thus, we can again obtain the description (19) for the cluster-mass distribution of A^m species ($m = 1,2,\ldots,n-1$) in the long-time limit, where $C_{1,m}$ in Eq. (19) is substituted by $C_{11,1}$ in this case. Meanwhile, the evolution of A^n species satisfies the modified scaling law and its cluster-mass distribution can be described by Eq. (20), where $C_{1,n}$ in Eq. (20) is substituted by $C_{11,n}$. Thus, for this case, we find the similar results to those in Sec. II A.

When $(2J-I_e)I_1A_{10} = (2J-I_n)I_eA_{n0}$, $b_2 = b_4 = 0$. We obtain the exact solutions of $\alpha_1(t)$ and $\alpha_n(t)$,

$$\alpha_1(t) = C_{13,1}t^{I_e(2J-I_n)/(4J^2-I_eI_n)},$$

$$\alpha_n(t) = C_{13,n}t^{I_n(2J-I_e)/(4J^2-I_eI_n)}, \tag{72}$$

where

$$C_{13,1} = \{[(4J^2-I_eI_n)/I_e(2J-I_n)]b_1^{2J/(2J-I_n)}\}^{I_e(2J-I_n)/(4J^2-I_eI_n)}$$

and

$$C_{13,n} = \{[(4J^2-I_eI_n)/I_n(2J-I_e)]b_3^{2J/(2J-I_e)}\}^{I_e(2J-I_n)/(4J^2-I_eI_n)}.$$

It is obvious that Eqs. (72) are similar to Eqs. (40) and (41) in Sec. II C, respectively. Thus, we find that each species has the conventional scaling description for its cluster-mass distribution in the long-time limit. One can easily obtain the same conventional scaling descriptions (42) for this case, where the constants $C_{4,m}$ and $C_{4,n}$ in Eqs. (42) are substituted by $C_{13,1}$ and $C_{13,n}$, respectively. Meanwhile, the exponents are the same as Eqs. (44). However, when $I_e > I_n$, A^m clusters ($m = 1,2,\ldots,n-1$) dominate over A^n clusters in the long-time limit, and when $I_e < I_n$, A^n clusters dominate over A^m clusters. These are just the contrary to those in Sec. II C.

When $(2J-I_e)I_1A_{10} < (2J-I_n)I_eA_{n0}$, we find that $\alpha_n \to \infty$ at $t \to \infty$. From Eq. (68), we obtain the asymptotic solution of $\alpha_n(t)$ in the long-time limit,

$$\alpha_n(t) \simeq C_{14,n}t, \tag{73}$$

where $C_{14,n} = b_4^{2J/(2J-I_e)}$. The asymptotic solution of $\alpha_1(t)$ can then be determined,

$$\alpha_1(t) \simeq C_{14,1} - C_{15,1}t^{1-2J/I_n}, \tag{74}$$

where $C_{14,1} = (I_nA_{n0}/2C_{14,n})^{I_e/2J}$ and $C_{15,1} = [I_1I_nA_{10}/2(2J-I_n)]C_{14,n}^{-2J/I_n}$. Equations (73) and (74) are similar to Eqs. (61) and (62) in Sec. II G, respectively. Thus, A^n species has the conventional description (63) for its cluster-mass distribution, where the constant $C_{9,n}$ in Eq. (63) is substituted by $C_{14,n}$ for this case. Meanwhile, we obtain the modified description (64) for A^m species ($m = 1,2,\ldots,n-1$), where the constant $C_{9,m}$ and $C_{10,m}$ are substituted by $C_{14,1}$ and $C_{15,1}$, respectively. The results of this case are identical with those in Sec. II G.

III. SUMMARY

We have studied an irreversible aggregation-annihilation model with n types of distinct species on the basis of the mean-field theory. Considering the constant-reaction-rate model, we analyze the kinetic behavior of the irreversible aggregation processes with jointly complete annihilation. Here, A^n species is assumed to be annihilated jointly by all the other A^m species ($m = 1,2,\ldots,n-1$). We introduce an equivalent aggregation rate for all A^m species, $I_e = (\Sigma_{1 \leqslant m \leqslant n-1}I_m^{-1})^{-1}$, which represents the contributions of all A^m species to the kinetics of the processes. It is found that the evolution behavior of the system depends crucially on the ratios of the rates I_n and I_e to the annihilation rate J. In several cases with different rate ratios, we obtain distinct evolution results of the system, which are illustrated in Table II. Here, it is noted that $\eta = (2J-I_e)I_1A_{10}/(2J-I_n)I_eA_{n0}$.

In short, the evolution behavior of the *n*-species aggregation process with joint annihilation always obeys a conventional scaling law or a modified one. The governing exponents depend strongly on the reaction rates for most cases, and for some special cases the initial concentrations also play an important role in the exponents. Moreover, there is at most one class of the species (A^n species or A^m species) that can survive finally in the steady state.

ACKNOWLEDGMENT

The project was supported by Zhejiang Provincial Natural Science Foundation of China under Grant No. 199050.

[1] S. K. Friedlander, *Smoke, Dust and Haze: Fundamentals of Aerosol Behavior* (Wiley, New York, 1977).
[2] F. Leyvraz and H.R. Tschudi, J. Phys. A **15**, 1951 (1982).
[3] M.H. Ernst, R.M. Ziff, and E.M. Hendriks, J. Colloid Interface Sci. **97**, 233 (1984).
[4] P. Meakin, Rep. Prog. Phys. **55**, 157 (1992).
[5] P. Meakin, Phys. Rev. Lett. **51**, 1119 (1983); M. Kolb, R. Botet, and R. Jullien, *ibid.* **51**, 1123 (1983).
[6] T. Vicsek and F. Family, Phys. Rev. Lett. **52**, 1669 (1984); T. Vicsek, P. Meakin, and F. Family, Phys. Rev. A **32**, 1122 (1985).
[7] K. Kang and S. Redner, Phys. Rev. A **30**, 2833 (1984).
[8] P. Meakin, T. Vicsek, and F. Family, Phys. Rev. B **31**, 564 (1985).
[9] F. Family and P. Meakin, Phys. Rev. A **40**, 3836 (1989).
[10] S. Song and D. Poland, Phys. Rev. A **46**, 5063 (1992).
[11] P.L. Krapivsky and E. Ben-Naim, Phys. Rev. E **53**, 291 (1996); P.L. Krapivsky and S. Redner, *ibid.* **54**, 3553 (1996).
[12] L. Frachebourg, Phys. Rev. Lett. **82**, 1502 (1999).
[13] K. Kang and S. Redner, Phys. Rev. Lett. **52**, 955 (1984); Phys. Rev. A **32**, 435 (1985).
[14] F. Leyvraz and S. Redner, Phys. Rev. Lett. **57**, 163 (1986); Phys. Rev. A **36**, 4033 (1987).
[15] M. Bramson and J.L. Lebowitz, Phys. Rev. Lett. **61**, 2397 (1988).
[16] E. Clement, L.M. Sander, and R. Kopelman, Phys. Rev. A **39**, 6455 (1989); **39**, 6466 (1989).
[17] I. Ispolatov and P.L. Krapivsky, Phys. Rev. E **53**, 3154 (1996); B. Bonnier and R. Brown, *ibid.* **55**, 6661 (1997).
[18] A.A. Ovchinnikov and Y.B. Zeldovich, Chem. Phys. **28**, 215 (1978).
[19] D. Toussaint and F. Wilczek, J. Chem. Phys. **78**, 2642 (1983).
[20] P.L. Krapivsky, Physica A **198**, 135 (1993); **198**, 150 (1993).
[21] I.M. Sokolov and A. Blumen, Phys. Rev. E **50**, 2335 (1994).
[22] D. Balboni, P.A. Rey, and M. Droz, Phys. Rev. E **52**, 6220 (1995).
[23] E. Ben-Naim and P.L. Krapivsky, Phys. Rev. E **52**, 6066 (1995).
[24] Ligen Zhang and Z.R. Yang, Physica A **237**, 441 (1997); Phys. Rev. E **55**, 1442 (1997).
[25] Jianhong Ke and Zhenquan Lin, Phys. Rev. E **65**, 051107 (2002).
[26] Jianhong Ke and Zhenquan Lin, Commun. Theor. Phys. (to be published).
[27] D. Ben-Avraham and S. Redner, Phys. Rev. A **34**, 501 (1986).